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Interpolation and Approximation Properties of Rational Coordinates over Quadrilaterals*

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This is a study of the properties of rational coordinate functions for the purposes of interpolation and approximation of functions over an arbitrary convex quadrilateral in the plane. In particular, we investigate the properties of the space of all real homogeneous polynomials in the rational coordinate functions. We show that the class $\mathscr{B}_n$ of all real homogeneous polynomials of degree n has the dimension $(n + 1)^2$ and contains the set of all real polynomials of degree n or less in the Cartesian coordinates. We construct a monomial basis for $\mathscr{B}_n$, and a canonical basis for Lagrange interpolation which can be used in finite element approximations. Finally, we define an approximation procedure in Sobolev spaces and derive estimates for the norms of the error function.

I. INTRODUCTION

In Refs. [1, 2], Wachspress has developed a method for constructing rational coordinates over arbitrary convex polygons in the plane. These rational coordinates are generalizations of the areal coordinates defined over a triangle. In the present study, we investigate the properties of these rational coordinates in the case of an arbitrary convex quadrilateral, Q. In particular, we are interested in their properties for the purposes of interpolation and approximation of functions.

In Section II, we first recall the definition of the rational coordinates, $\{w_i: i = 1, 2, 3, 4\}$, and their properties, as given by Wachspress. We then prove a new nonlinear relation among the w_i's (Lemma 2(ii)), as well as a new property of each individual w_i (Lemma 3). Using a more or less natural coordinate system, we subsequently establish an explicit representation for each w_i.

In Section III, we introduce the class of all real homogeneous polynomials in the rational coordinates on Q. This class is dense in the space of all real continuous functions on Q. We show that the class $\mathscr{B}_n$ of all real homogeneous polynomials of degree n in the variables (w_1, w_2, w_3, w_4) on Q has

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the dimension $(n + 1)^2$, and that it contains the set $\mathscr{P}_n$ of all real polynomials of degree n or less in the Cartesian coordinates x and y. In fact, $\mathscr{B}_n$ may be viewed as the natural generalization to the case of an arbitrary quadrilateral of, on the one hand, the space of all real homogeneous polynomials of degree n in the areal coordinates defined over a triangle, and, on the other hand, the tensor product space of all real polynomials of degree n in the Cartesian coordinates defined over a rectangle.

In Section IV, we construct a monomial basis for $\mathscr{B}_n$, and in Section V a cardinal basis for Lagrange interpolation, suitable for use in finite element approximation over a domain which has been partitioned in quadrilaterals in some arbitrary manner. Finally, in Section VI, we define an approximation procedure in Sobolev spaces and derive estimates for the norms of the error function.

II. Wachspress' Rational Coordinate Functions

Consider an arbitrary convex quadrilateral Q in the extended plane. Let its vertices be labeled P_1, P_2, P_3, P_4. For $i = 1, 2, 3, 4$, let the line containing the segment P_iP_{i+1} be given by the linear equation $l_{i+1} = 0$. (Here, we have adopted the convention that indices on P, l, and later also on w, shall *always* be taken modulo 4.) We denote the two external diagonal points of Q by S and T, with $S = (l_2 = 0) \cap (l_4 = 0)$ and $T = (l_1 = 0) \cap (l_3 = 0)$. Let the line containing the segment ST be given by the linear equation $m = 0$ (see Fig. 1).

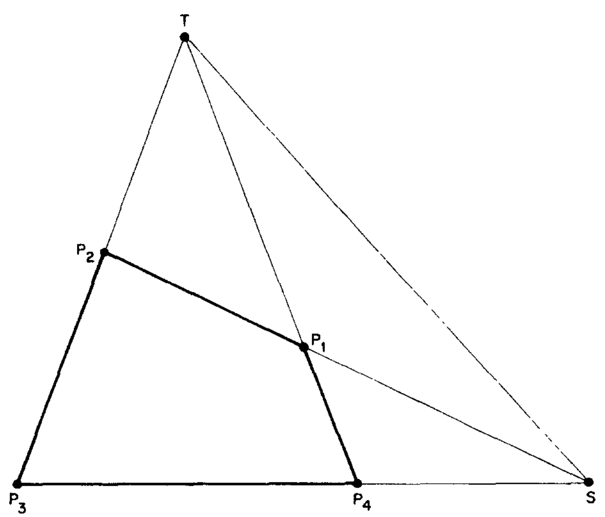

Fig. 1. A convex quadrilateral $(P_1P_2P_3P_4)$ with its external diagonal points S and T.

Following Wachspress, we introduce a set of four rational coordinate functions associated with Q, as follows.

$$\{w_i : i = 1, 2, 3, 4\}, \qquad \text{with} \qquad w_i = \frac{m(P_i)}{l_{i-1}(P_i)\, l_{i+2}(P_i)} \frac{l_{i-1} l_{i+2}}{m},$$

where $m(P_i)$, $l_{i-1}(P_i)$, and $l_{i+2}(P_i)$ denote the values of the linear forms m, l_{i-1}, and l_{i+2}, respectively, at the point P_i. In the case of a rectangle, $m = 0$ is the equation of the line at infinity, and we take $w_i = l_{i-1}l_{i-2}/[l_{i-1}(P_i)l_{i-2}(P_i)]$.

LEMMA 1. *For each i ($i = 1, 2, 3, 4$), w_i has the following properties.* (i) *w_i is infinitely differentiable inside Q;* (ii) $w_i(P_i) = 1$; (iii) *$w_i = 0$ on $P_{i+1}P_{i+2}$ and $P_{i-2}P_{i-1}$ (the two sides of Q opposite the vertex P_i);* (iv) *w_i varies linearly along P_iP_{i+1} and $P_{i-1}P_i$ (the two sides of Q adjacent to the vertex P_i);* (v) *$w_i > 0$ at all points inside Q.*

These properties have been proved by Wachspress in Ref. [1].

The following lemma expresses two properties of the set $\{w_i : i = 1, 2, 3, 4\}$.

LEMMA 2. *The four functions w_i, which make up the set $\{w_i : i = 1, 2, 3, 4\}$, satisfy the following relations.* (i) $w_1 + w_2 + w_3 + w_4 = 1$; (ii) $w_1w_3/w_2w_4 = |SP_2||SP_4|/|SP_1||SP_3|$, *where $|SP_i|$ denotes the (undirected) distance from S to P_i.*

Proof. (i) This property has been proved by Wachspress in Ref. [1]. (ii) To prove the nonlinear redundancy relation, we observe that, from the definition of w_i, we have

$$\frac{w_1w_3}{w_2w_4} = \frac{m(P_1)}{m(P_4)} \frac{m(P_3)}{m(P_2)} \frac{l_1(P_2)}{l_1(P_3)} \frac{l_2(P_4)}{l_2(P_3)} \frac{l_3(P_4)}{l_3(P_1)} \frac{l_4(P_2)}{l_4(P_1)}.$$

Since $m(P_1)$ is proportional to the distance from P_1 to the line $m = 0$, and $m(P_4)$ is proportional to the distance from P_4 to the line $m = 0$, with the same proportionality constant, the ratio $m(P_1)/m(P_4)$ is equal to the ratio $|TP_1|/|TP_4|$, where $|TP_1|$ and $|TP_4|$ are the distances from T to P_1 and T to P_4, respectively. The other ratios in the expression for w_1w_3/w_2w_4 above can be replaced in a similar way. The result is

$$\frac{w_1w_3}{w_2w_4} = \frac{|TP_1|}{|TP_4|} \frac{|TP_3|}{|TP_2|} \frac{|TP_2|}{|TP_3|} \frac{|SP_4|}{|SP_3|} \frac{|TP_4|}{|TP_1|} \frac{|SP_2|}{|SP_1|} = \frac{|SP_4|}{|SP_3|} \frac{|SP_2|}{|SP_1|}.$$

The next lemma is a more complete statement of Lemma 1, property (iv).

LEMMA 3. *For each i ($i = 1, 2, 3, 4$), w_i varies linearly along any line through either of the exterior diagonal points S and T.*

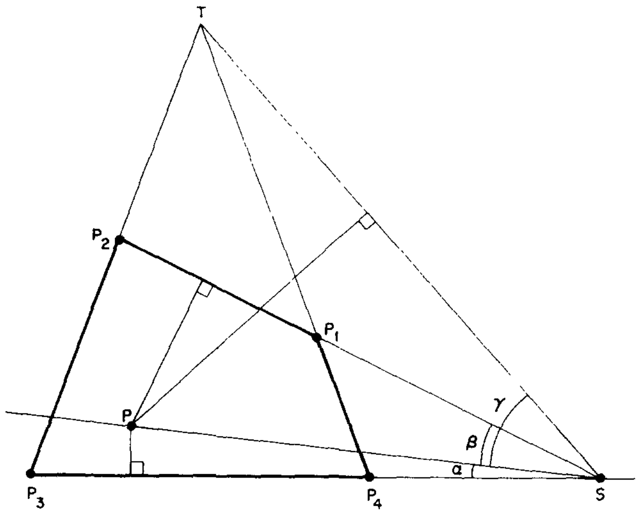

FIG. 2. The point P on the line $l = 0$ through the external diagonal point S.

Proof. Consider any line $l = 0$ through S (see Fig. 2). For a point P on this line we have

$$w_1(P) = \frac{m(P_1)}{l_4(P_1)\, l_3(P_1)} \frac{l_4(P)}{m(P)} l_3(P).$$

Since $l_4(P)$ and $m(P)$ are proportional to the perpendicular distances from P to the lines $l_4 = 0$ and $m = 0$, respectively, with different, but constant, proportionality constants, the ratio $l_4(P)/m(P)$ is, apart from a constant factor, equal to the ratio $\sin \alpha/\sin \gamma$, which is independent of P. Hence, the ratio $l_4(P)/m(P)$ does not vary along the line $l = 0$; consequently, it can be replaced by its value, for example, at S. Thus, along any line through S we have

$$w_1 = \frac{m(P_1)}{l_4(P_1)\, l_3(P_1)} \frac{l_4(S)}{m(S)} l_3 .$$

In the same way we show that

$$w_2 = \frac{m(P_2)}{l_1(P_2)\, l_4(P_2)} \frac{l_4(S)}{m(S)} l_1 ,$$

$$w_3 = \frac{m(P_3)}{l_2(P_3)\, l_1(P_3)} \frac{l_2(S)}{m(S)} l_1 ,$$

$$w_4 = \frac{m(P_4)}{l_3(P_4)\, l_2(P_4)} \frac{l_2(S)}{m(S)} l_3 .$$

A similar argument is used for any line through T.

We now derive an explicit representation for the function w_i, using a more or less natural coordinate system in Q.

First, we introduce the normalized barycentric or areal coordinates $(\zeta_1, \zeta_2, \zeta_3)$ relative to the reference triangle P_3ST. The ζ's are linear functions of x and y; the coefficients depend on the coordinates of the vertices P_3, S, and T. In terms of the areal coordinates we define the new coordinates s and t,

$$s = \zeta_2/(\zeta_1 + \zeta_2), \qquad t = \zeta_3/(\zeta_1 + \zeta_3). \tag{1}$$

Thus, the line through $T(0, 0, 1)$ and an arbitrary point $P(\zeta_1, \zeta_2, \zeta_3)$ intersects the side P_3S at the point with areal coordinates $(1 - s, s, 0)$. Similarly, the line through $S(0, 1, 0)$ and P intersects the side P_3T at the point with areal coordinates $(1 - t, 0, t)$. In other words, s increases along P_3P_4 from the value 0 at P_3 to a value σ, $\sigma < 1$, at P_4, and t increases along P_3P_2 from the value 0 at P_3 to a value τ, $\tau < 1$, at P_2. The quadrilateral Q in the (x, y)-plane is thus mapped onto the rectangle $\hat{Q} = \{(s, t)\colon 0 \leqslant s \leqslant \sigma, 0 \leqslant t \leqslant \tau\}$ in the (s, t)-plane.

The transformation which is inverse to the transformation (1) is easily found by means of identity $\zeta_1 + \zeta_2 + \zeta_3 = 1$, together with Eq. (1),

$$\zeta_1 = \frac{(1 - s)(1 - t)}{1 - st}, \qquad \zeta_2 = \frac{s(1 - t)}{1 - st}, \qquad \zeta_3 = \frac{t(1 - s)}{1 - st}. \tag{2}$$

In terms of the areal coordinates $(\zeta_1, \zeta_2, \zeta_3)$ or the (s, t)-coordinates, the equations of the four sides of Q and of the line through S and T, are

$$\begin{aligned}
P_4P_1\colon&\quad l_1 \equiv \sigma\zeta_1 - (1 - \sigma)\,\zeta_2 \equiv (\sigma - s)(1 - t)/(1 - st) = 0,\\
P_1P_2\colon&\quad l_2 \equiv \tau\zeta_1 - (1 - \tau)\,\zeta_3 \equiv (1 - s)(\tau - t)/(1 - st) = 0,\\
P_2P_3\colon&\quad l_3 \equiv \zeta_2 \equiv s(1 - t)/(1 - st) = 0,\\
P_3P_4\colon&\quad l_4 \equiv \zeta_3 \equiv t(1 - s)/(1 - st) = 0,\\
ST\colon&\quad m \equiv \zeta_1 \equiv (1 - s)(1 - t)/(1 - st) = 0.
\end{aligned}$$

The (s, t)-coordinates of the vertices of Q are

$$P_1\colon (\sigma, \tau), \qquad P_2\colon (0, \tau), \qquad P_3\colon (0, 0), \qquad P_4\colon (\sigma, 0).$$

From these data and the definition of the coordinate functions w_i we obtain the representations

$$w_1 \equiv w_1(s, t) = [(1 - \sigma\tau)/\sigma\tau][st/(1 - st)], \tag{3a}$$

$$w_2 \equiv w_2(s, t) = (1/\sigma\tau)[t(\sigma - s)/(1 - st)], \tag{3b}$$

$$w_3 \equiv w_3(s, t) = (1/\sigma\tau)[(\sigma - s)(\tau - t)/(1 - st)], \tag{3c}$$

$$w_4 \equiv w_4(s, t) = (1/\sigma\tau)[s(\tau - t)/(1 - st)]. \tag{3d}$$

In the (s, t)-coordinate system, the nonlinear relationship which was established in Lemma 2(ii), becomes

$$w_1 w_3 / w_2 w_4 = 1 - \sigma\tau. \tag{4}$$

Finally, we give the representation of the areal coordinate $(\zeta_1, \zeta_2, \zeta_3)$ in terms of the coordinate functions w_i,

$$\zeta_1 = [(1-\sigma)(1-\tau)/(1-\sigma\tau)]\, w_1 + (1-\tau)\, w_2 + w_3 + (1-\sigma)\, w_4, \tag{5a}$$
$$\zeta_2 = [\sigma(1-\tau)/(1-\sigma\tau)]\, w_1 + \sigma w_4, \tag{5b}$$
$$\zeta_3 = [\tau(1-\sigma)/(1-\sigma\tau)]\, w_1 + \tau w_2. \tag{5c}$$

Thus, the areal coordinates are homogeneous linear functions of the rational coordinates.

III. The Spaces $\mathscr{B}_n(Q)$

Wachspress has indicated how the rational coordinate functions w_i can be utilized for the purpose of approximating functions defined over a general quadrilateral Q by collocation at points on the boundary of Q. Here, we take a different approach and study the approximating properties of the class of all real homogeneous polynomials in the variables w_1, w_2, w_3, and w_4.

Theorem 1. *For any convex quadrilateral Q, the class of all real, homogeneous polynomials in the rational coordinates (w_1, w_2, w_3, w_4) is dense in the space of real continuous functions on Q.*

Proof. On the basis of Lemma 3 it is easy to show that $w_i(P) \neq w_i(P')$ for at least one index i, whenever P and P' are two distinct points inside Q. The theorem is then an immediate consequence of the Stone approximation theorem (see Ref. [3, Chap. 1, Section 4]).

Let $\mathscr{B}_n = \mathscr{B}_n(Q)$ denote the linear space of all real homogeneous polynomials of degree n $(n = 0, 1,...)$ in the variables w_1, w_2, w_3, w_4.

Theorem 2. *$\mathscr{B}_n$ is a finite-dimensional subspace of the space of all real continuous functions on Q; its dimension is $(n+1)^2$.*

Proof. Because of Lemma 2(i), we can map the coordinate set $\{w_i: i = 1, 2, 3, 4\}$ onto another set $\{\phi_i: i = 1, 2, 3, 4\}$ which contains the unit element,

$$\begin{aligned}
\phi_1 &= w_1 + w_2 + w_3 + w_4, \\
\phi_2 &= (1 - st)^{1/2} \left(\frac{w_1}{1-\sigma\tau} + w_4\right), \\
\phi_3 &= (1 - st)^{1/2} \left(\frac{w_1}{1-\sigma\tau} + w_2\right), \\
\phi_4 &= \frac{w_1}{1-\sigma\tau},
\end{aligned}$$

with the inverse mapping given by

$$\begin{aligned}
w_1 &= (1 - \sigma\tau)\,\phi_4\,,\\
w_2 &= [\phi_3/(1 - st)^{1/2}] - \phi_4\,,\\
w_3 &= \phi_1 - \frac{\phi_2}{(1 - st)^{1/2}} - \frac{\phi_3}{(1 - st)^{1/2}} + (1 + \sigma\tau)\,\phi_4\,,\\
w_4 &= [\phi_2/(1 - st)^{1/2}] - \phi_4\,.
\end{aligned}$$

The mapping is nonsingular:

$$\partial(\phi_1, \phi_2, \phi_3, \phi_4)/\partial(w_1, w_2, w_3, w_4) = (1 - st)/(1 - \sigma\tau),$$

which is nonzero in $\hat{Q}$. In terms of the variables s and t, the new coordinates have the representation

$$\begin{aligned}
\phi_1 &= 1,\\
\phi_2 &= s/\sigma(1 - st)^{1/2},\\
\phi_3 &= t/\tau(1 - st)^{1/2},\\
\phi_4 &= st/\sigma\tau(1 - st),
\end{aligned}$$

from which we immediately conclude that the ϕ's satisfy the nonlinear relation

$$\phi_4 = \phi_2\phi_3\,.$$

Now, consider an arbitrary element $f \in \mathscr{B}_n$. It has the form

$$f = \sum_{(\alpha)} a_\alpha w_1^{\alpha_1} w_2^{\alpha_2} w_3^{\alpha_3} w_4^{\alpha_4},$$

where the sum extends over all combinations $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = n$; the real coefficients a_α are independent of the w's. Since each w_i is a homogeneous linear function of ϕ_1, $\phi_2/(1 - st)^{1/2}$, $\phi_3/(1 - st)^{1/2}$, and ϕ_4, we can rewrite f in the form

$$f = \sum_{(\beta)} b_\beta \phi_1^{\beta_1} \phi_2^{\beta_2} \phi_3^{\beta_3} \phi_4^{\beta_4} / [(1 - st)^{1/2}]^{\beta_2 + \beta_3},$$

where, now, the sum extends over all combinations $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ with $|\beta| = n$. If we express the ϕ's in terms of the variables s and t, we obtain

$$f = \sum_{(\beta)} b_\beta (s/\sigma)^{\beta_2 + \beta_4} (t/\tau)^{\beta_3 + \beta_4} / (1 - st)^{\beta_2 + \beta_3 + \beta_4}.$$

Again, the sum extends over all combinations $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ with $|\beta| = n$. The latter expression can, in turn, be rewritten in the form

$$f = \frac{1}{(1 - st)^n} \sum_{(\beta)} b_\beta (s/\sigma)^{\beta_2 + \beta_4} (t/\tau)^{\beta_3 + \beta_4} (1 - st)^{\beta_1},$$

or, after a rearrangement of terms,

$$f = \frac{1}{(1 - st)^n} \sum_{l=0}^{n} \sum_{m=0}^{n} c_{lm} s^l t^m.$$

Since the set of monomials $\{s^l t^m: l = 0,..., n; m = 0,..., n\}$ is linearly independent over the domain $\hat{Q}$, it follows that the dimension of the linear space $\mathscr{B}_n$ is the same as the dimension of $\mathrm{sp}\{s^l t^m: l = 0,..., n; m = 0,..., n\}$. The latter has dimension $(n + 1)^2$, so

$$\dim(\mathscr{B}_n) = (n + 1)^2.$$

The finite-dimensional subspaces $\mathscr{B}_n$ provide a convenient mechanism for approximating functions in a finite element procedure. Generally, the relevant question in this connection is, What classes of polynomials in the variables x and y are contained in the spaces $\mathscr{B}_n$?

Let $\mathscr{P}_n = \mathscr{P}_n(Q)$ denote the linear space of all functions which are defined on Q and are represented there by a real polynomial of degree less than or equal to n ($n = 0, 1,...$) in the variables x and y.

THEOREM 3. $\mathscr{P}_n(Q) \subset \mathscr{B}_n(Q)$ *for* $n = 0, 1,...$.

Proof. Any element of $\mathscr{P}_n$ can be represented on Q either as a polynomial of degree at most n in the variables x and y, or as a homogeneous polynomial of degree n in the areal coordinates $\zeta_1, \zeta_2, \zeta_3$. Since each ζ_i is a homogeneous linear function of the rational coordinates (w_1, w_2, w_3, w_4) (see Eq. (5) of the previous section), it follows that any element of $\mathscr{P}_n$ can also be represented as a homogeneous polynomial of degree n in the variables (w_1, w_2, w_3, w_4). That is, any element of $\mathscr{P}_n$ corresponds uniquely to an element of $\mathscr{B}_n$. Hence, $\mathscr{P}_n \subset \mathscr{B}_n$.

It is worthwhile to investigate what happens to the space $\mathscr{B}_n(Q)$, when, on the one hand, Q degenerates into a triangle, and, on the other hand, Q is a rectangle.

For the sake of definiteness, let us assume that the distance from the vertex P_1 to the diagonal P_2P_4 (see Fig. 1) is decreased continuously. The external diagonal points S and T then move along the lines $l_4 = 0$ and $l_2 = 0$ toward the vertices P_4 and P_2, respectively. In the limit, as Q coincides with the triangle $P_3P_4P_2$, the reference triangle P_3ST is the same as this triangle, so the areal coordinates $(\zeta_1, \zeta_2, \zeta_3)$ become the areal coordinates relative to Q itself. Furthermore, in the (s, t)-plane, the image $\hat{Q}$ of Q coincides with the unit square, $\hat{Q} = \{(s, t): 0 \leqslant s \leqslant 1, 0 \leqslant t \leqslant 1\}$. From Eq. (3) we see that the coordinate function w_1 tends to zero, while w_2, w_3, and w_4 tend to $t(1-s)/(1-st) = \zeta_3$, $(1-s)(1-t)/(1-st) = \zeta_1$, and $s(1-t)/(1-st) = \zeta_2$, respectively. Hence, as Q degenerates into a triangle, the space $\mathscr{B}_n(Q)$ becomes the space of all real homogeneous polynomials of degree n in the areal coordinates relative to Q.

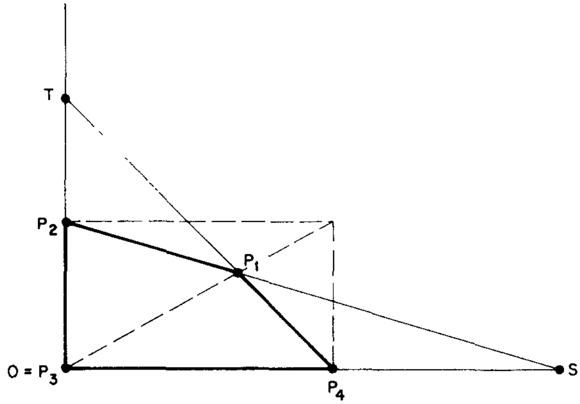

FIG. 3. The quadrilateral Q with right angle at P_3.

On the other hand, consider the quadrilateral Q of Fig. 3. The (x, y)-coordinates of its vertices are: $P_1(\alpha a, \alpha b)$, $P_2(0, b)$, $P_3(0, 0)$, $P_4(a, 0)$, with α constant, $0 < \alpha < 1$. As $\alpha \to 1$, P_1 moves up along the line $y = (b/a)x$, and in the limit, $\alpha = 1$, Q becomes the rectangle $Q = \{(x, y); 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b\}$. One readily verifies that the (x, y)-coordinates of the external diagonal points S and T are $(\alpha a/(1 - \alpha), 0)$ and $(0, \alpha b/(1 - \alpha))$, respectively. Relative to the triangle P_3ST, the areal coordinates are

$$\zeta_1 = 1 - \frac{1-\alpha}{\alpha}\frac{x}{a} - \frac{1-\alpha}{\alpha}\frac{y}{b}, \qquad \zeta_2 = \frac{1-\alpha}{\alpha}\frac{x}{a}, \qquad \zeta_3 = \frac{1-\alpha}{\alpha}\frac{y}{b},$$

so

$$s = \frac{\zeta_2}{\zeta_1 + \zeta_2} = \frac{1-\alpha}{\alpha}\frac{x}{a}\Big/\left(1 - \frac{1-\alpha}{\alpha}\frac{y}{b}\right), \qquad \text{with} \quad 0 \leqslant s \leqslant \sigma = \frac{1-\alpha}{\alpha},$$

$$t = \frac{\zeta_3}{\zeta_1 + \zeta_3} = \frac{1-\alpha}{\alpha}\frac{y}{b}\Big/\left(1 - \frac{1-\alpha}{\alpha}\frac{x}{a}\right), \qquad \text{with} \quad 0 \leqslant t \leqslant \tau = \frac{1-\alpha}{\alpha}.$$

Now, as $\alpha \to 1$, the image $\hat{Q}$ of Q in the (s, t)-plane shrinks to a single point at the origin. From Eq. (3) we see that, in the limit $\alpha = 1$,

$$w_1 = \frac{x}{a}\frac{y}{b}, \qquad w_2 = \frac{y}{b}\left(1 - \frac{x}{a}\right), \qquad w_3 = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right), \qquad w_4 = \frac{x}{a}\left(1 - \frac{y}{b}\right).$$

Hence, as Q becomes a rectangle in the plane, the space $\mathscr{B}_n(Q)$ becomes the space of all real polynomials of degree at most n in the variables x and y, i.e., $\mathscr{B}_n(Q) = \mathscr{P}_n(Q)$.

From these two limiting cases we conclude that the space $\mathscr{B}_n(Q)$ represents a natural generalization to the case of an arbitrary quadrilateral Q in the plane of, on the one hand, the space of all real homogeneous polynomials of degree n in the areal coordinates defined over a triangle, and, on the other hand, the tensor product space of all real polynomials of degree n in the Cartesian coordinates defined over a rectangle.

IV. Construction of a Monomial Basis for $\mathscr{B}_n(Q)$

In this section we construct a basis for the finite-dimensional subspace $\mathscr{B}_n = \mathscr{B}_n(Q)$, which consists of monomials of degree n in the variables w_1, w_2, w_3, and w_4. The construction proceeds by induction on n.

1. $n = 0$. A monomial basis for $\mathscr{B}_0$ is, obviously,

$$\Omega_0 = \{\omega_1^{(0)}\}, \qquad \text{with} \qquad \omega_1^{(0)} = 1.$$

2. $n = 1$. A monomial basis for $\mathscr{B}_1$ is

$$\Omega_1 = \{\omega_i^{(1)}: i = 1, 2, 3, 4\}, \qquad \text{with} \qquad \omega_i^{(1)} = w_i .$$

3. $n = 2, 3,\ldots$. Assume that we have found a monomial basis for $\mathscr{B}_{n-2}$. Let this basis be denoted by Ω_{n-2},

$$\Omega_{n-2} = \{\omega_i^{(n-2)}: i = 1,\ldots, (n-1)^2\}.$$

Now, form the set of monomials

$$Z_n = \{w_i^{\,n}, w_i^{n-1}w_{i+1} ,\ldots, w_i w_{i+1}^{n-1} : i = 1, 2, 3, 4\}.$$

There are a total of $4n$ elements in the set Z_n, and each element is a monomial of degree n. The set is linearly independent on the boundary of Q. Let the elements in the set be linearly ordered and denoted by $z_i^{(n)}$, so that

$$Z_n = \{z_i^{(n)}: i = 1,\ldots, 4n\}.$$

Then, define

$$\begin{aligned} \omega_i^{(n)} &= z_i^{(n)}, & i &= 1,\ldots, 4n, \\ \omega_i^{(n)} &= w_2 w_4 \omega_{i-4n}^{(n-2)}, & i &= 4n+1,\ldots, (n+1)^2. \end{aligned}$$

The set

$$\Omega_n = \{\omega_i^{(n)}: i = 1,..., (n+1)^2\}$$

thus consists of monomials of degree n. If we can show that this set is linearly independent, it follows that Ω_n is a monomial basis for $\mathscr{B}_n$.

Suppose there exists a set of constants $\{\alpha_i: i = 1,..., (n+1)^2\}$ such that

$$\sum_{i=1}^{(n+1)^2} \alpha_i \omega_i^{(n)} = 0 \quad \text{in} \quad Q. \tag{6}$$

Since $w_2 w_4 = 0$ on the boundary of Q, all $\omega_i^{(n)}$ with $i > 4n$ vanish identically on the boundary of Q, and Eq. (6) implies

$$\sum_{i=1}^{4n} \alpha_i \omega_i^{(n)} = 0$$

on the boundary of Q. But, since the set Z_n is linearly independent on the boundary of Q, the latter relation in turn implies that all α_i with $i \leqslant 4n$ are zero. Hence, the relation (6) above reduces to

$$\sum_{i=4n+1}^{(n+1)^2} \alpha_i \omega_i^{(n)} = 0 \quad \text{in} \quad Q,$$

or, since $\omega_i^{(n)} = w_2 w_4 \omega_{i-4n}^{(n-2)}$ for $i = 4n+1,..., (n+1)^2$, and $w_2 w_4 > 0$ in Q,

$$\sum_{i=1}^{(n-1)^2} \alpha_{4n+i} \omega_i^{(n-2)} = 0 \quad \text{in} \quad Q. \tag{7}$$

By continuity, this same relation then holds for all of Q, including the boundary. But, since Ω_{n-2} is, by assumption, a monomial basis for $\mathscr{B}_{n-2}$, Eq. (7) can be satisfied only if all coefficients α_i with $i > 4n$ are zero. In other words, all the coefficients in Eq. (6) vanish; i.e., the set Ω_n is linearly independent and, therefore, forms a monomial basis for $\mathscr{B}_n$.

V. Lagrange Interpolation over a Quadrilateral Mesh

The rational coordinate functions may be used conveniently for the numerical solution of boundary value problems by finite element methods. In this section, we indicate how one can construct a polynomial basis in the rational coordinates for Lagrange interpolation over an arbitrary quadrilateral. We assume that this quadrilateral is part of a network of quadrilaterals, and that the interpolating polynomial function must be matched across interelement boundaries at specified sets of points, to meet the global continuity requirements appropriate to the particular problem under investigation.

Consider an arbitrary quadrilateral, Q, with its associated rational coordinate functions (w_1, w_2, w_3, w_4). For a fixed positive integer n, define the linear space $\mathscr{B}_n = \mathscr{B}_n(Q)$ as before, and let $\Omega_n = \{\omega_i^{(n)}: i = 1,..., (n+1)^2\}$ be its monomial basis. We restrict our discussion to nondeficient elements, so that the number of nodes associated with Q is equal to the dimension of $\mathscr{B}_n$, viz $(n+1)^2$. These nodes are distributed over Q in the following way.

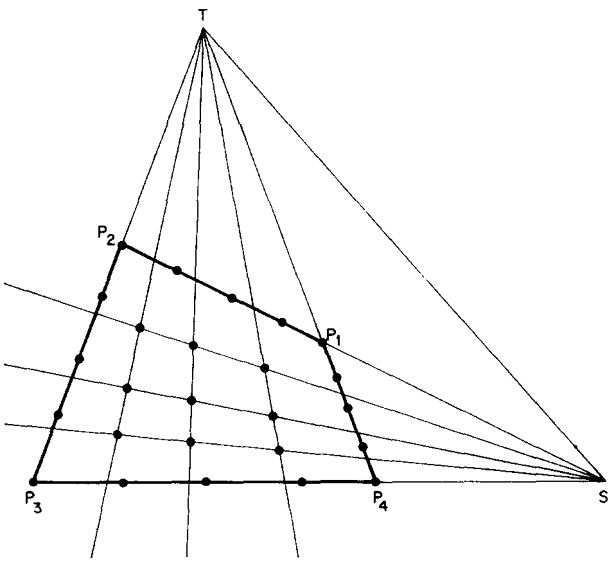

FIG. 4. A grid for Lagrange interpolation over the quadrilateral $P_1P_2P_3P_4$.

Four vertex nodes $\{P_i: i = 1, 2, 3, 4\}$, $4(n-1)$ side nodes $\{P_i: i = 5,..., 4n\}$, and $(n-1)^2$ interior nodes $\{P_i: i = 4n+1,..., (n+1)^2\}$ (see Fig. 4). The side nodes may, but need not, evenly subdivide the sides of Q. The interior nodes may be taken, for example, at the lattice of $(n-1)^2$ points $X_S \cap X_T$, where X_S is a pencil of $n-1$ lines through S,

$$X_S = \left\{ w_2 = \frac{t_k}{\tau - t_k} w_3 : k = 1,..., n-1 \right\},$$

$$\text{with} \quad 0 < t_1 < t_2 \cdots < t_{n-1} < \tau,$$

and X_T is a pencil of $n-1$ lines through T,

$$X_T = \left\{ w_4 = \frac{s_k}{\sigma - s_k} w_3 : k = 1,\ldots, n-1 \right\},$$
$$\text{with} \quad 0 < s_1 < s_2 \cdots < s_{n-1} < \sigma.$$

Notice that the side nodes generally do not line up with the interior nodes.

Now, any element $p \in \mathscr{B}_n$ has a unique representation,

$$p = \sum_{i=1}^{(n+1)^2} \alpha_i \omega_i^{(n)}. \tag{8}$$

It is our objective to represent p in terms of its nodal values $\{p_j = p(P_j): j = 1,\ldots, (n+1)^2\}$; thus

$$p = \sum_{i=1}^{(n+1)^2} p_j \phi_j^{(n)}. \tag{9}$$

To this end, we evaluate Eq. (8) at the nodes $\{P_j : j = 1,\ldots, (n+1)^2\}$. The result is a system of linear equations for the coefficients $\{\alpha_i: i = 1,\ldots, (n+1)^2\}$,

$$\sum_{i=1}^{(n+1)^2} \alpha_i \omega_{ij}^{(n)} = p_j, \qquad j = 1,\ldots, (n+1)^2,$$

where $\omega_{ij}^{(n)}$ is the value of $\omega_i^{(n)}$ at P_j. Since each $\omega_i^{(n)}$ with $i > 4n$ vanishes identically on the boundary of Q, the coefficient matrix $\Omega = (\omega_{ij}^{(n)})$ can be partitioned,

$$\Omega = \begin{pmatrix} \Omega_{11} & 0 \\ \Omega_{21} & \Omega_{22} \end{pmatrix},$$

with Ω_{11} a $4n \times 4n$ matrix and Ω_{22} a $(n-1)^2 \times (n-1)^2$ matrix. Both Ω_{11} and Ω_{22} are nonsingular, so Ω^{-1} exists and is given by

$$\Omega^{-1} = \begin{pmatrix} \Omega_{11}^{-1} & 0 \\ -\Omega_{22}^{-1}\Omega_{21}\Omega_{11}^{-1} & \Omega_{22}^{-1} \end{pmatrix}.$$

Thus, if we denote the (i,j)-element of Ω^{-1} by ω_{ij}^{-1}, we have

$$\alpha_i = \sum_{j=1}^{(n+1)^2} \omega_{ij} p_j, \qquad i = 1,\ldots, (n+1)^2.$$

Substitution of this result in Eq. (8) gives

$$p = \sum_{i=1}^{(n+1)^2} \sum_{j=1}^{(n+1)^2} \omega_{ij}^{-1} p_j \omega_i^{(n)}.$$

This is a representation of the type (9), with

$$\phi_j^{(n)} = \sum_{i=1}^{(n+1)^2} \omega_{ij}^{-1} \omega_i^{(n)}, \qquad j = 1,..., (n+1)^2.$$

The set of polynomials

$$\Phi_n = \{\phi_j^{(n)}: j = 1,..., (n+1)^2\}$$

forms a canonical basis for $\mathscr{B}_n$ for Lagrange interpolation over the set of nodes $\{P_j: j = 1,..., (n+1)^2\}$, i.e.,

$$\phi_j^{(n)}(P_i) = \delta_{ij} \qquad \text{for} \quad i, j = 1,..., (n+1)^2.$$

VI. Approximation in Sobolev Spaces

The set of interpolatory polynomials Φ_n obtained in the previous section can be used to define an approximation procedure for functions on Q. Approximation of a function u over Q is achieved through a projection operator Π_n into the finite-dimensional subspace $\mathscr{B}_n(Q)$,

$$\Pi_n : u \to \hat{u} = \Pi_n u = \sum_{i=1}^{(n+1)^2} u_i \phi_i^{(n)}, \qquad u_i = u(P_i).$$

We observe that the set $\mathscr{P}_n$ of polynomials of degree at most n in the (x, y)-variables is invariant under Π_n,

$$\Pi_n u = u \qquad \text{for all} \quad u \in \mathscr{P}_n = \mathscr{P}_n(Q).$$

For approximation of functions in Sobolev spaces, we can apply a result of Ciarlet and Raviart [4], to obtain estimates for the Sobolev norm of the error $u - \Pi_n u$. These estimates involve the following two geometric parameters related to Q,

h = diameter of Q,

$\rho = \sup\{$diameter of the inscribed circles in $Q\}$.

We recall that $W^{l,p}(Q)$, for any integer l, $l \geqslant 1$, and any p, $1 \leqslant p \leqslant \infty$, is the Sobolev space of all (equivalence classes of) real-valued functions which, together with their generalized partial derivatives of order $\leqslant l$, belong to $L^p(Q)$. The norm $\| \ \|_{l,p}$ and seminorm $| \ |_{l,p}$ in $W^{l,p}(Q)$ are defined by

$$\| u \|_{l,p} = \left(\sum_{k=0}^{l} \| D^k u \|_p^p \right)^{1/p},$$

$$| u |_{l,p} = \| D^l u \|_p,$$

respectively, for $1 \leqslant p < \infty$, and

$$\| u \|_{l,\infty} = \max\{\| D^k u \|_\infty : k = 0,..., l\},$$
$$| u |_{l,\infty} = \| D^l u \|_\infty ,$$

respectively, for $p = \infty$.

THEOREM 4. *Let p be given, $1 \leqslant p \leqslant \infty$; let $n \geqslant 0$ be a fixed integer, and let l be an integer with $0 \leqslant l \leqslant n + 1$. For any $u \in W^{n+1,p}(Q)$ (and for h sufficiently small if $p < \infty$), the error $u - \Pi_n u$ satisfies the estimate*

$$\| u - \Pi_n u \|_{l,p} \leqslant C \, | u |_{n+1,p} (h^{n+1}/\rho^l),$$

with C a constant, which is independent of u.

The theorem is a direct consequence of Ref. [4], Theorem 5.

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